# On the type of attractor ruin of GCMs found by the clustering algorithm

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**Abstract**

In this talk, we consider the intermittency of the globally coupled chaotic maps (Kaneko, 1990). In a certain parameter region, this system exhibits intermittency, a nonlinear phenomenon in which it switches between periodic-like and chaotic states. Mierski and Pilarczyk (2025) proposed a method to detect attractor ruin in intermittent trajectories using a clustering algorithm. However, the details did not show the property of the phenomenon. We report the distribution characteristics of the residence time during the intermittent phase. In particular, this system exhibits, in a sense, “strong” or “weak” attractor ruin, and we discuss the possibility that the domain of initial values can be classified based on the dynamics on these attractor ruins.

## 1 Introduction

Nonlinear phenomena, including chaos, commonly emerge in complex systems such as neural networks [20], ecological networks [11], chemical oscillators [15], laser systems [22], and climate networks [3], and understanding these phenomena remains an important challenge across multiple disciplines.

Kaneko introduced a discrete dynamical system called the globally coupled map (GCMs) as a simple model for high-dimensional chaotic coupled systems [6]. Classically, dynamics were investigated from a clustering viewpoint using an indicator called the effective dimension, corresponding to the number of clusters at an affective precision [5, 7, 8, 9, 13, 1]. Analytically, for example, Maistrenko et al. have shown the stability of periodic cluster states [18, 17, 16, 19, 12].

In particular, it is well known that intermittent behavior occurs in specific parameter regions of GCMs. GCMs have parameter regimes known as the glassy phase, in which supertransients are observed, and the intermittent phase, in which intermittency occurs. In the glassy phase, analysis of the the effective dimension reveals periods of temporary low-dimensional stability between supertransitient events. The phenomenon in which a system irregularly alternates between chaotic high-dimensional states and metastable low-dimensional states is referred to as chaotic itinerancy [21, 10]. This behavior is regarded as a universal phenomenon in high-dimensional dynamical systems and has attracted considerable attention.

However, in the intermittent phase, the effective dimension remains largely unchanged, making it difficult to distinguish between the laminar and burst phase of intermittency. This limitation left open the question of how to interpret the low-dimensional states, known as attractor ruins.

Recently, Mierski and Pilarczyk proposed a method using the hierarchical clustering algorithm, Hierarchical Density-Based Spatial Clustering of Applications with Noise (HDBSCAN

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[2]), to identify attractor ruins from intermittent trajectories [14]. They examined whether the transitions between attractor ruins revealed by clustering constitute chaotic itinerancy through statistical tests.

Nevertheless, they did not investigate quantities such as residence time distributions, which characterize the durations for which trajectories remain temporarily trapped in metastable states and are central to intermittency analysis.

Therefore, we investigated the residence time distribution, focusing particularly on its power-law strength. Specifically, we generated finite-length trajectories using a computer and examined their properties. We found that, depending on the initial conditions, the power strength can depend on the length of the trajectory.

The structure of this paper is as follows. Section 2 briefly introduces the model and its main characteristics. Section 3 outlines the clustering-based method proposed by Mierski and Pilarczyk. Our main results are presented in Section 4, and Section 5 provides concluding remarks.

## 2 Model

GCMs is the high-dimensional dynamical system coupled chaotic elements, and it is defined as follows:

$$x_{t+1}(i) = (1-\varepsilon) f(x_t(i)) + \frac{\varepsilon}{d} \sum_{j=1}^{d} f(x_t(j)), \quad i = 1, 2, \cdots, d$$

$$f(x) = \alpha x(1-x), \quad \boldsymbol{x}_t = (x_t(1), \ldots, x_t(d)) \in \mathbb{R}^d, \quad t \in \{0\} \cup \mathbb{N}, \quad (\alpha, \varepsilon) \in \mathbb{R}^2.$$

This represents a general parameter setting; hereafter, we strict $(\alpha, \varepsilon) \in [0,4] \times [0,1]$ and consider the phase space to be $[0,1]^d$, in accordance with the properties of the logistic map.

When $x(i)$ and $x(j)$ satisfy $|x(i) - x(j)| < \delta$, they are considered to be synchronized with accuracy $\delta$ and thus belong to the same cluster.

It is known that the stability of the cluster state depends on the coupling strength $\varepsilon$ and the nonlinearity $\alpha$, and the parameter region is classified as follows depending on the dynamics of the dynamical system:

- **Coherent phase** : On the strong coupling strength and the weak nonlinearity parameter, the system converges to a fully synchronized state (coherent state), and the model behaves as a one-dimensional (logistic) map.
- **Ordered phase** : The system converges to a state of some number of clusters. In other words, GCM can be viewed as a low-dimensional dynamical system. Furthermore, the dynamics are periodic. It is also known that the final cluster state changes depending on the initial conditions.
- **Glassy phase** : It is known that in the region corresponding to the critical curves of the blowout bifurcation and Hopf bifurcation, the transient process depends heavily on the initial conditions. In particular, a phenomenon called a supertransients, which is an extremely long transient state, is observed.
- **Turbulent phase** : For strong nonlinearity and weak coupling, the system does not synchronize across all elements $x(i)$. Strictly speaking, however, the turbulent phase region contains interspersed bands of other phases, such as periodic windows.

- **Intermittent phase** : This parameter region corresponds to the destabilization of the periodic state observed in the ordered phase, leading to repeated irregular transitions (intermittent behavior) mediated by chaotic states. An example of an orbit is shown in Figure 1.

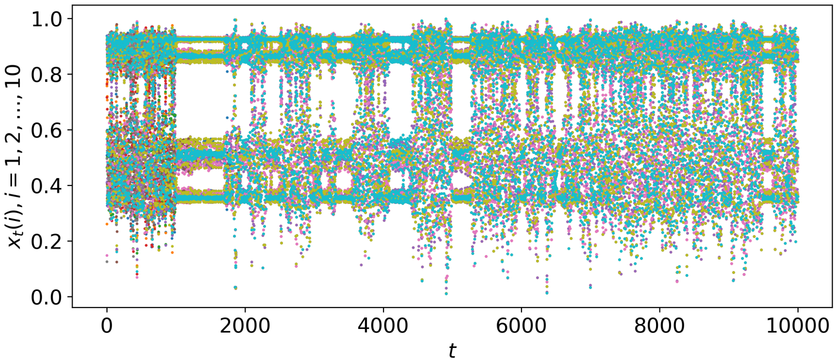


Figure 1: Example of an orbit in the intermittent phase. The parameters were set to $d = 10$, $\alpha = 4.0$, and $\epsilon = 0.213$, with initial conditions $\boldsymbol{x}_0$ sampled uniformly from $[0, 1]$.

## 3 Method

In this section, we present a clustering-based method to distinguish, in a certain sense, low-dimensional states from chaotic high-dimensional states using a given orbit $\{\boldsymbol{x}_t\}$. The method is extremely simple: plot a single orbit in phase space and cluster it as a point cloud. Specifically, we apply HDBSCAN to the point cloud obtained from the orbit. HDBSCAN is an optimization algorithm that extracts dense regions within data as clusters, excluding outliers and noise as the remainder. It features only one hyperparameter, called the minimum cluster size, and does not specify the number of clusters like $k$-means clustering.

Mierski and Pilarczyk represented transitions between clusters using transition matrices after clustering. They then reconfigured the clusters from a periodicity perspective by observing these inter-cluster transitions. In other words, the low-dimensional state here corresponds to periodic dynamics.

As an example, Figure 2 shows the entire orbit $\{\boldsymbol{x}_t\}$ in phase space together with the results of reconfiguring clusters using HDBSCAN, where clusters are indicated by colored points and residual noise by gray points. Following previous work, the clustering hyperparameter is set to $M = 300$.

Figure 3 shows a diagram of these clustered points arranged in a time series for each cluster.

Mierski and Pilarczyk examined representative values for the residence time in each attractor ruins and noise series. After applying modifications to the series based on these values, they tested for autocorrelation, non-stationarity, and randomness using the Ljung-Box test, augmented Dickey-Fuller test, and O'Brien-Dyck runs test, respectively, to determine whether chaotic itinerancy was occurring.

In the next section, we show that in the intermittent phase, the residence time distribution follows a power-law distribution, and that its power depends on the initial conditions and the length of the observed orbit.

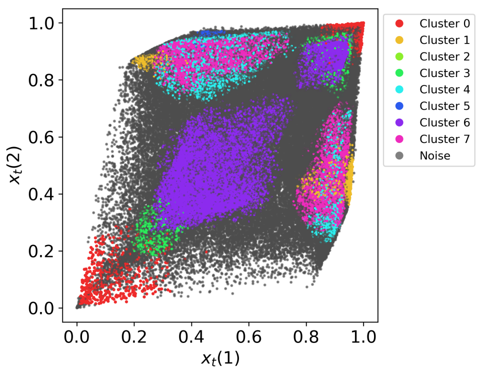


Figure 2: The clustered orbit where $d = 10$, $\alpha = 3.8$, and $\varepsilon = 0.213$. In this case, they are classified into 8 clusters. Note that this figure projects onto a plane defined by the first and second coordinates.

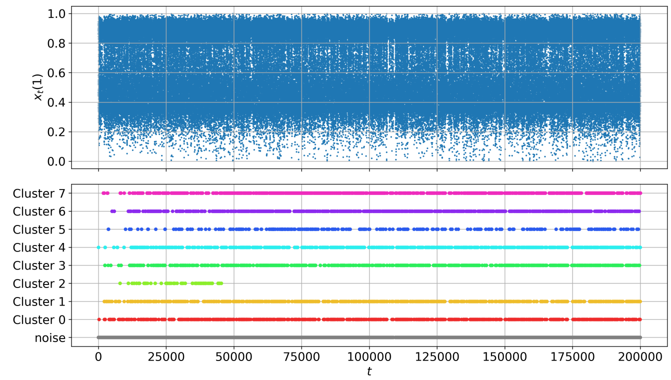


Figure 3: The upper panel shows the time series of the orbits themselves; for clarity, only the first coordinate is displayed. The lower panel presents the time series corresponding to each identified cluster.

# 4 Main Results

Let be $M = 300$, system size $d = 10$ and length of orbit $N = 200,000$. The nonlinear parameter and coupling strength are set to $\alpha = 4.0$ and $\varepsilon = 0.213$, respectively, corresponding to the intermittent phase.

Figure 4 is the residence time distribution for each classified cluster. Figure 5 is the residence time distribution for all clusters. In other words, it is distribution obtained by summing the frequencies for each residence time in Figure 4. This result suggests that the residence times follow a power-law distribution. That is, it indicates an extremely high number of short stays in each cluster and a small number of long stays. The power (orange line) when viewed as a power-law distribution is $-1.504\cdots$. Note that this line was obtained using linear regression based solely on data for small residence times.

While the results in Figure 5 pertain to a single orbit $\{\boldsymbol{x}_t\}$, it is interesting to examine how this power changes with the initial conditions. Next, we present the results of investigating the dependence of the power on the initial conditions and the observation time (length of orbit).

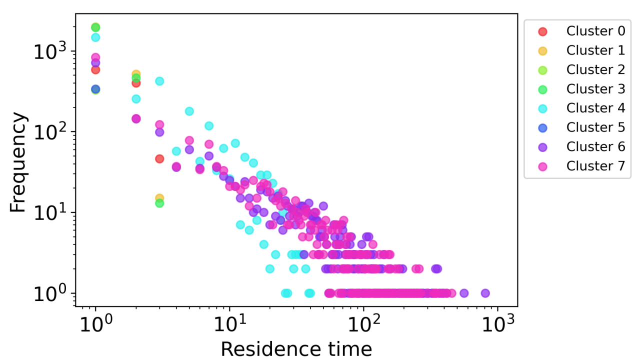


Figure 4: The residence time distribution for each clusters where $d = 10$, $\alpha = 3.8$, and $\varepsilon = 0.213$. Each color corresponds to the respective time series in Figure 3.

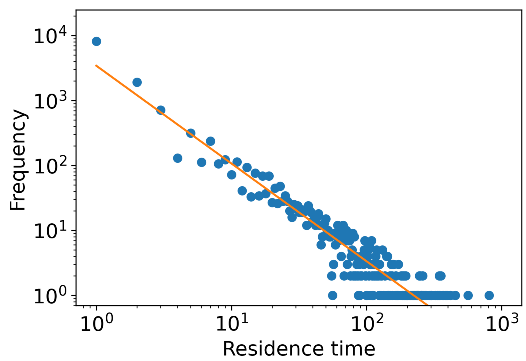


Figure 5: The residence time distribution for all clusters. The power (orange line) when viewed as a power-law distribution is $-1.504\cdots$.

Figure 6 shows the power-law exponents of the residence time distributions obtained by applying clustering each orbit for 1,000 randomly sampled initial conditions. The orbit length $T$ was increased in steps of 5,000 from 100,000 to 200,000.

For most initial conditions (95.5%, shown in blue in Fig.6), the power exponent lies between -1.75 and -1.25 and essentially independent of the orbit length, suggesting slow convergence.

In addition, a small fraction of cases (1.8%, shown in orange in Fig.6) exhibits smaller power exponents that are also independent of the orbit length. These cases are characterized by the absence of long residence times, indicating that the attractor ruins identified by clustering are “weakly” stable in a certain sense.

Furthermore, cases in which the power exponent gradually increases with the orbit length were observed (1.2%, shown in green in Fig.6). For short orbits, “weak” attractor ruins are detected (see Fig.7), whereas sufficiently long orbits reveal attractor ruins that attract most initial conditions (see Fig.8). The data are classified into 11 clusters and noise in both cases. However, examining the cluster transitions reveals that after approximately 110,000, some states cease to appear, leaving only a limited set of states. In other words, a phenomenon analogous to supertransitions emerges in the perspective of clustering.

Finally, a small number of initial conditions (1.5%, shown in red in Fig.6) exhibit a strong dependence of the power exponent on the orbit length, accompanied by a clustering phase transition.

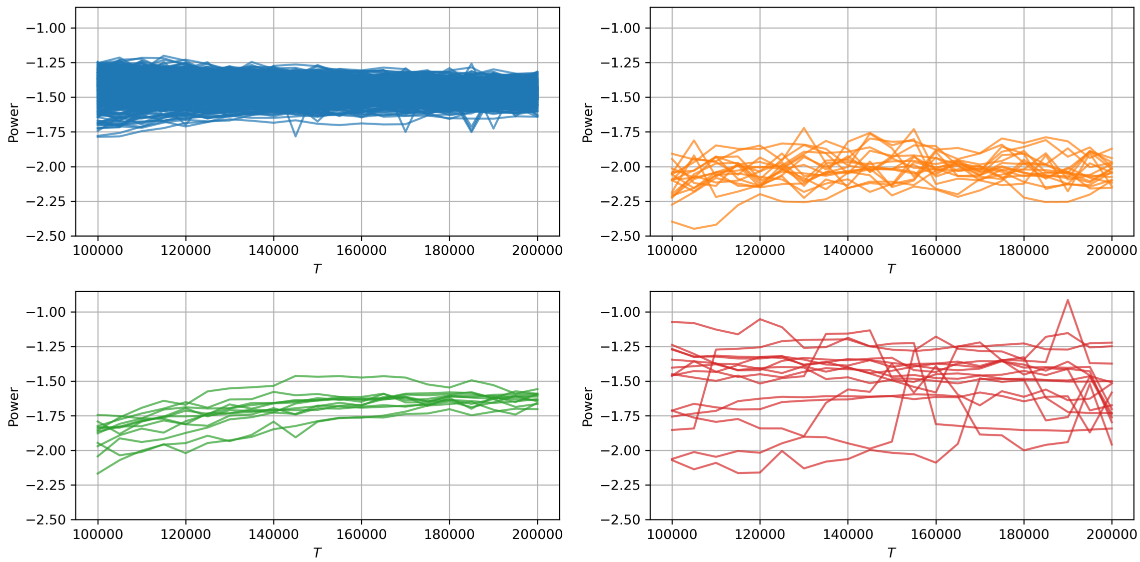


Figure 6: Dependence of the power exponents on the orbit length $T$ for 1,000 different initial condisions. The parameters are set to $d = 10$, $\alpha = 4.0$, $\varepsilon = 0.213$, and $M = 300$.

## 5 Conclusion

In this paper, we identified attractor ruins appearing intermittency in GCMs using clustering methods. We characterized the distribution of residence times in these attractor ruins using power-law exponents and showed that this distribution depends on both the initial conditions and the observation time. Specifically, we suggested that initial conditions can potentially be classified into four patterns based on the power-law exponent.

1. The power exponent lies between -1.75 and -1.25 for most initial conditions (95.5%, shown in blue in Fig.6).

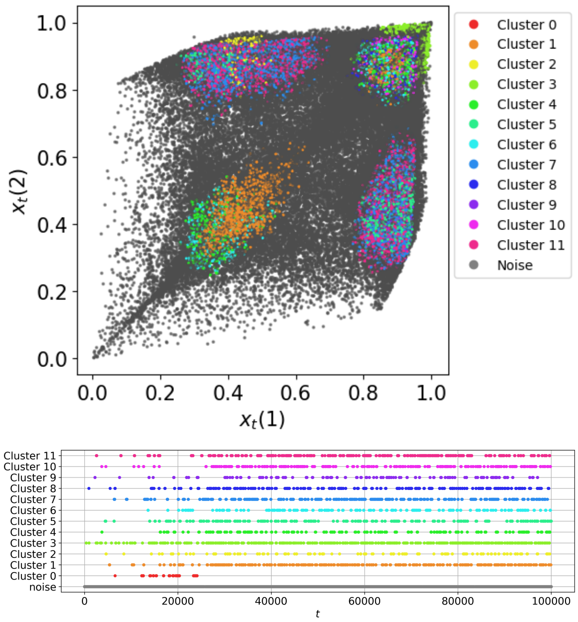


Figure 7: Clustering result for orbit length $T = 100,000$ (top) and clustering transitions (bottom). the power at this case is $-1.831\cdots$.

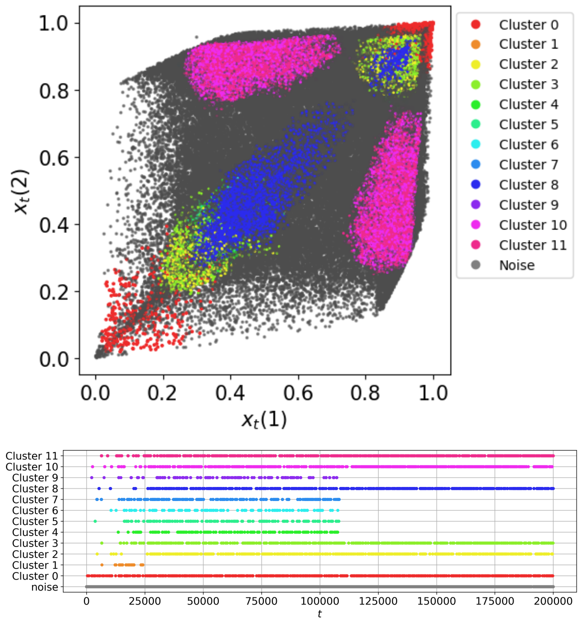


Figure 8: Clustering result for orbit length $T = 200,000$ (top) and clustering transitions (bottom). The power at this case is $-1.630\cdots$.

2. The smaller power exponents that are also independent of the orbit length (1.8%, shown in orange in Fig.6).

3. The power exponent gradually increases with orbit length were observed (1.2%, shown in green in Fig.6).

4. The strong dependence of the power exponent on the orbit length, accompanied by a clustering phase transition (1.5%, shown in red in Fig.6).

In particular, for the first case, the average of the power-law exponent at an orbit length of $T = 200,000$ is approximately $-1.457$. This value is close to the exponent $-1.5$ characterizing the laminar residence-time distribution in on-off intermittency [4], which has been suggested to be related to chaotic itinerancy in GCMs. From this perspective, our results may provide numerical evidence that chaotic itinerancy in the intermittent phase of GCMs originates from on-off intermittency.

Nevertheless, the underlying mechanism and related details remain unclear and require further investigation. Moreover, analyses based on clustering methods face practical limitations, as their application to high-dimensional systems is computationally demanding. Resolving these issues is essential for improving our understanding of high-dimensional dynamical systems and constitutes an important challenge for future research.

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